

One-dimensional matter waves as a multi-state bit

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We design a technique to control the position of a localized matter wave. Our system is composed by a two counter-phased periodic potentials and a third optical lattice which can be chosen to be either periodic or disordered. The only control needed on the system is a three-state switch that allows the instantaneous selection of the desired potential. We show that this framework is robust and the multi-state bit behavior can be observed under generic hypothesis.

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I. INTRODUCTION

Nowadays Bose Einstein condensates (BEC) [1–4] are routinely used in combination with optical potentials in order to have a direct access to the fundamental quantum behaviors on a macroscopic scale. The state of art in this field offers a wide range of possibilities in terms of manipulation over these systems [5–23] and there is a deep knowledge of the expected behaviors from the theoretical side.

The dimensionality of the system can be reduced by flattening the BEC (effective 2D system [14]) or elongating it (effective 1D system [11–13]). In the 1D case, interesting boundary conditions can be realized: the elongated BEC can be trapped in a box [15], in a torus [33] or in an harmonic trap [9, 10], among other possibilities [7, 8].

Many different optical potentials can be realized for this system. Without any presumption of being exhaustive, we recall the possibility of generating both periodic [11–14] and disordered [16–26] lattices. The latter family of potentials has been employed in order to observe Anderson localization phenomena [16, 17, 21, 22, 26]. 1D speckle potential in particular have been the object of an intensive study in recent years, both from theoretical and experimental side. The localization properties of a speckle system have been investigated both in infinitely extended [17, 18] and box bounded systems [25, 26], showing that the finite length case can have an even stronger degree of localization compared with the infinite length case, under the proper conditions [26]. In addition to the wide selection of feasible optical potentials, we recall the recent possibility of painting an arbitrary shape time-averaged optical dipole potential [7].

Finally, the Fano-Feshbach resonances [30–32] can be employed to lessen or even eliminate the non linear effects of the self-interaction, leading the dynamics of the system to be ruled by a linear Schrödinger equation.

This strong degree of control over a quantum system allows for the research of technological applications. In particular, investigations in using matter wave as quantum switches or quantum information device has been done in recent years [36, 37]. In this article we propose a general technique to employ a 1D BEC, either self-interacting or not, as a multi-state bit, by a proper

temporal alternation of three optical potentials. This design is completely new to the best of our knowledge and it is the first example of BEC used as a classical multi-state bit. The proposed technique is robust and can be applied under a range of different specifications, both in the box and the torus cases. Also the number of states is an arbitrary choice.

The article is organized as follows. In Sec. II we define the general features of our system. In Sec. III we discuss the way to employ this system as a multi-state bit. The robustness of the system is discussed in Sec. IV, where different possible implementations are compared. Our results are summarized in Sec. V.

II. MODEL AND METHODS

Let us consider a non interacting matter wave in a 1D optical potential which can be selected amongst three possible choices. The system is finite and its length is L . The Hamiltonian of the system can be written in a dimensionless form as

$$\hat{H} = -\frac{d^2}{dx^2} + \sum_{k=1}^3 \mathbf{1}_{\{k=c\}} v_k(x, s_k), \quad (1)$$

where $c \in \{1, 2, 3\}$ is the external choice of the potential and v_i are three potentials described below. We consider also a zeroth case ($c = 0$) which is not selectable during the time evolution of the system but it is used just to set the initial conditions. The dimensionless hamiltonian in Eq. (1) is rescaled by an energy value $E_\xi = \frac{\hbar^2}{2m\xi^2}$ with $\xi \simeq 1\mu m$. E_ξ is related to $v_1(x, s_1)$ and defined together with it. We want to study the dynamics of this system, under the hypothesis that c can be changed instantaneously. This is reasonable considering that $v_k(x, s_k)$ can be optically generated. When considering the time evolution and the presence of self-interaction, the system is fully described by the Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi = \left[\hat{H} + 2\alpha\beta\frac{|\psi|^2}{\sigma^2} + \alpha(\sigma^2 + \sigma^{-2}) \right] \psi, \quad (2)$$

where $\tau = E_\xi t/\hbar$ is a rescaled dimensionless time and $\sigma^2(x, \tau) = \sqrt{1 + \beta|\psi(x, \tau)|^2}$. α and β are defined and

fully specified in Appendix A. The nonlinear terms describe the self interaction of the Bose Einstein Condensate (BEC) that can be used in order to realize the system. Eq. (2) is an effective 1D model known as non-polynomial Schrödinger equation (NPSE) [35]. This is obtained from the Gross-Pitaevskii 3D equation [34] in order to provide an approximate description for the BEC dynamics under radial confinement. Nowadays, the self interaction can be chosen to be repulsive ($\beta > 0$) [28, 29], attractive ($\beta < 0$) [30, 31] or absent ($\beta = 0$) [32], depending on the experimental settings. The considered potentials are defined in the following.

(v_1) In Section III we explain how v_1 can be employed in order to keep $|\psi(x, \tau)|^2$ stable over time. To this end, we consider two possibilities: a disordered potential v_1^d and a periodic potential v_1^p .

(v_1^d) We consider an optical speckle $v_1^d(x) = V_0 v(x/\xi)$, with intensity $V_0 = \langle v_1 \rangle$ and autocorrelation length ξ [24, 27]. The probability distribution of $v(x)$ is e^{-v} . Moreover it holds that

$$\langle v(y)v(y+x) \rangle_y = 1 + \text{sinc}^2\left(\frac{x}{\xi}\right)$$

Optical speckle is obtained by transmission of a laser beam through a medium with a random phase profile, such as a ground glass plate. The resulting complex electric field is a sum of independent random variables and forms a Gaussian process. Atoms experience a random potential proportional to the intensity of the field. V_0 can be either positive or negative, the potential resulting in a series of barriers or wells. However, in both cases it is possible to observe Anderson localization phenomena [25, 26]. The autocorrelation length ξ represents a natural scale for the system and $E_\xi = \hbar^2/2m\xi^2$ is the corresponding energy scale. We define

$$v_1^d(x, s_1) = s_1 v(x) \quad (3)$$

where $s_1 = V_0/E_\xi$ is a rescaled dimensionless intensity. The speckle pattern can be generated numerically as discussed in [24] (and references therein).

(v_1^p) This potential must be smooth and periodic:

$$v_1^p(x, s_1) = s_1 f\left(\text{mod}\left(x, \frac{\Delta}{2}\right)\right), \quad (4)$$

where $\Delta = L/N$ ($N \in \mathbb{N}$), $f(\Delta/2+x) = f(\Delta/2-x)$ and $df(x)/dx = 0 \Leftrightarrow \text{mod}(x, \Delta/2) = 0$. $\text{mod}(a, b)$ is the remainder of the division a/b .

In section IV we consider $v_1^p(x, s_1) = s_1 \cos(\frac{4\pi}{\Delta}x)$ as a realistic case.

(v_2) v_2 can be obtained from v_1^p by doubling the period and considering a different amplitude s_2 , which is a parameter independent from s_1 .

$$v_2(x, s_2) = s_2 f(\text{mod}(x, \Delta)), \quad (5)$$

where the same requirements described above holds. In Section IV we consider $v_2(x, s_1) = s_2 \cos(\frac{2\pi}{\Delta}x)$ as a realistic case.

(v_3) Also the third potential is smooth and periodic, in antiphase with $v_2(x, s_2)$:

$$v_3(x, s_3) = s_3 f\left(\text{mod}(x, \Delta) + \frac{\Delta}{2}\right) \quad (6)$$

In the following we will always consider $s_2 = s_3$ only.

(v_0) The initial condition $\psi(x, \tau = 0)$ must be localized around x_0 such that $\text{mod}(x_0, \Delta) = \Delta/4$. This can be achieved forcing the BEC to the ground state of a properly chosen optical potential $v_0(x, s_0)$. In Section III we consider

$$v_0(x, s_0) = s_0 \cos\left(\frac{4\pi}{\Delta}x\right) + \omega^2 \left(x - \frac{L}{2} - \frac{\Delta}{4}\right)^2 \quad (7)$$

where ω^2 is a constant dimensioned as length^{-2} and valued as $|L^{-1}|$.

In the next section we show that the system described above acts as a multi-state bit under two alternative boundary conditions: box and torus.

A. Measure of the system stability

In Section III we describe a method to control the localization position x_{loc} of the matter wave ψ by changing the c value with a proper timing. Hence we are interested to prevent the spatial expansion of ψ , in order to be able to measure $x_{loc}(\tau)$ even for $\tau \gg 0$. The Participation Ratio (PR) is commonly used in literature as a measure of the localization degree [26, 38].

$$PR[\psi] = \frac{1}{L \int_L dx |\psi(x)|^4} \quad (8)$$

We introduce the following quantity in order to compare the PR value measured during the evolution of the system with the initial one.

$$DPR(\tau) = \frac{PR[\psi(x, \tau)]}{PR[\psi(x, 0)]} \quad (9)$$

The measure of $x_{loc}(\tau)$ becomes more difficult and less precise at increasing $DPR(\tau)$ values. In our system x_{loc} is clearly measurable when $DPR \lesssim 10$ while it cannot be defined nor observed anymore when $DPR \gtrsim 20$.

III. HOW TO USE THE SYSTEM AS A MULTI-STATE BIT

A multi-state bit can assume a state chosen from a discrete and finite set. We can conventionally define this

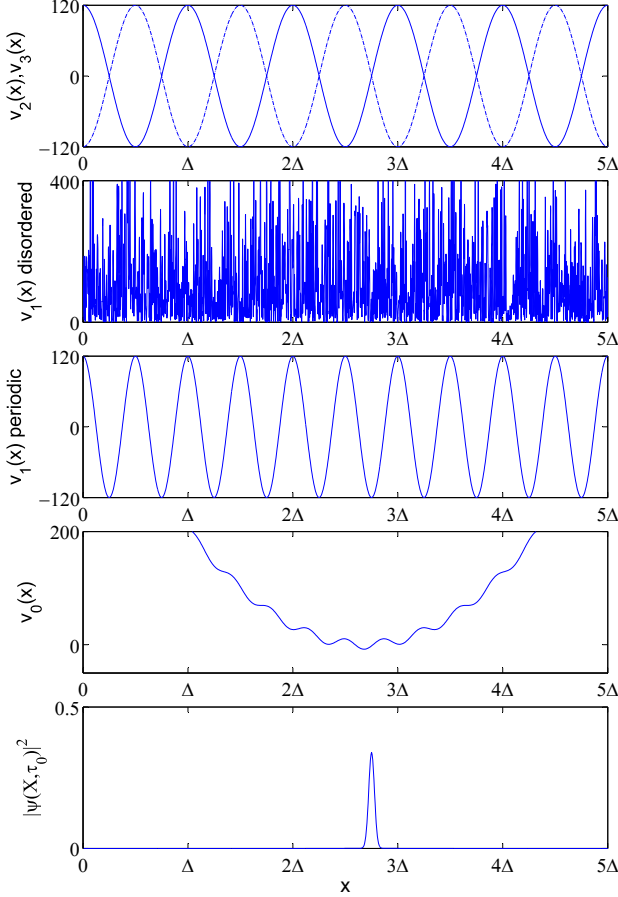


FIG. 1. A possible specification of the system described in Section II: $N = 5$ and $v_0 = v_1 = v_2 = v_3 = 120$. From the top to the bottom: the potentials v_2 (solid line) and v_3 (dotted line); the speckle potential v_1^d ; the periodic potential v_1^p ; the potential v_0 ; the initial density profile $|\psi(x, \tau_0)|^2$.

set of states by partitioning the system into $2N = 2L/\Delta$ intervals and labeling each interval with a number $n \in \{1, \dots, 2N\}$. We want to perform three basic operations on our system in order to consider it a true multi-state bit: writing information, keeping memory of it over an arbitrary time lapse, reading it again.

A. Reading information from the position of a localized matter wave

Using a matter wave allows us to measure the density profile directly. Since the chosen initial condition of the system is a localized state $\psi(x, \tau_0)$, the most of the density is concentrated in a small region. If $\text{DPR}(\tau) \lesssim 10$ while the system evolves, we can associate a number n_j

to every instant τ by measuring the density profile:

$$\varphi_R(\psi|\tau) = \sum_{j=1}^{2N} n_j \mathbf{1}_{\{j-1 \leq \frac{2x_{loc}(\tau)}{\Delta} < j\}} \quad (10)$$

with $x_{loc}(\tau) = \underset{x \in [0, L]}{\text{argmax}} |\psi(x, \tau)|^2$,

Eq. (10) allows us to read the information stored in our system. In the next paragraph we discuss how to write information in the system (using operators φ_+ and φ_-) and how to store the information over a time lapse δ keeping n_τ constant (using operator φ_δ). The convenient choice of n_j depends on the boundary conditions. In case of box boundary conditions we choose

$$n_j = \frac{j+1}{2} \mathbf{1}_{\{\text{mod}(j,2)=1\}} + \left(2N - \frac{j}{2} + 1\right) \mathbf{1}_{\{\text{mod}(j,2)=0\}} \quad (11)$$

In case of toroidal boundary conditions we choose

$$n_j = \frac{j + \text{mod}(j, 2)}{2} \quad (12)$$

We discuss the reason of this choices in paragraph III B 3. In our example ($N = 5$) eq. (11) leads to

$$\underline{n} = (1, 10, 2, 9, 3, 8, 4, 7, 5, 6)$$

and eq. (12) leads to

$$\underline{n} = (1, 1, 2, 2, 3, 3, 4, 4, 5, 5)$$

as shown in Figure 3.

B. Writing and maintaining information in the system

As discussed in Section II, the only way that we have to influence the system is switching c_τ from a value to another. We want to use this possibility to define three actions which affect the system as follows:

$$\varphi_R[\varphi_+(\psi|\tau)|\tau + \epsilon] = \varphi_R(\psi|\tau) + 1 \quad (13)$$

$$\varphi_R[\varphi_-(\psi|\tau)|\tau + \epsilon] = \varphi_R(\psi|\tau) - 1 \quad (14)$$

$$\varphi_R[\varphi_\delta(\psi|\tau)|\tau + \delta] = \varphi_R(\psi|\tau) \quad (15)$$

where ϵ is the time interval necessary to apply the operators φ_\pm and δ is a time interval over which the information has to be stored in the system. As explained in paragraph III B 3, the boundary conditions affect the definition of the φ_+ and φ_- operators. In case of box conditions we have

$$\varphi_R[\varphi_\pm(\psi|\tau)|\tau + \epsilon] = \text{mod}[\varphi_R(\psi|\tau) - 1 \pm 1, 2N] + 1 \quad (16)$$

In case of toroidal conditions we have

$$\varphi_R[\varphi_\pm(\psi|\tau)|\tau + \epsilon] = \text{mod}[\varphi_R(\psi|\tau) - 1 \pm 1, N] + 1 \quad (17)$$

1. Definition of φ_δ

The definition of $\varphi_\delta(\cdot|\tau)$ is based on different principles in case we use v_1^d or v_1^p .

In case we use a disordered potential v_1^d , it can cause the Anderson localization of the system and it inhibits any transport phenomena. Hence, provided that the disordered potential amplitude is big enough, any localized matter wave $\psi(x, \tau_0)$ should remain localized at the same position when observed in $\tau_0 + \delta$.

In case we use a periodic potential v_1^p , it can inhibit any transport phenomena too, provided that the localization point is exactly coincident with a local minimum of the potential and the amplitude is big enough.

In both cases we can define φ_δ as

$$\varphi_\delta(\psi|\tau_0) = \int_0^L \psi_{y,\tau_0} K(y, x, \tau_0, \tau_0 + \delta | c_\tau = 1) dy \quad (18)$$

where $K(y, x, \tau_0, \tau_0 + \delta)$ is the propagator associated with Eq. (2). Figure 3 (right panel) gives a graphical explanation of ϕ_δ when using v_1^p .

2. Definition of φ_\pm

The definition of $\varphi_\pm(\cdot|\tau)$ is based on the fact that a localized matter wave can experience a periodic potential as the single well where the mass is concentrated, provided that the potential amplitude is big enough and that x_{loc} is near enough to the local minimum x_{min} of the potential. In case of a symmetric well, the symmetry of the eigenstates is well defined and there is a time interval $\epsilon/2$ after which it holds that

$$\psi\left(x, \tau + \frac{\epsilon}{2}\right) \simeq \psi\left(x + 2(x_{min} - x_{loc}), \tau\right) \quad (19)$$

Let us suppose that

$$x_{loc}(\tau) = x_{min} - \frac{\Delta}{4} - \delta_x \quad \text{with } \delta_x \ll \Delta \quad (20)$$

. From Eq. (19) we have

$$x_{loc}\left(\tau + \frac{\epsilon}{2}\right) = 2x_{min} - x_{loc}(\tau) = x_{min} + \frac{\Delta}{4} + \delta_x \quad (21)$$

Applying an instantaneous π phase shift to the periodic potential leads to a displacement of the local minimum $x_{min} \rightarrow x'_{min} = x_{min} + \Delta/2$. Now we have

$$x_{loc}\left(\tau + \frac{\epsilon}{2}\right) = x'_{min} - \frac{\Delta}{4} + \delta_x \quad (22)$$

and after one more $\epsilon/2$ time lapse we obtain

$$x_{loc}(\tau + \epsilon) = 2x'_{min} - x_{loc}\left(\tau + \frac{\epsilon}{2}\right) = x_{loc}(\tau) + \Delta \quad (23)$$

So we made the localized matter wave to travel a distance Δ by applying two periodic potential in antiphase

and with big amplitude. In Section IV we investigate the conditions under which this displacement can be iterated preventing $DPR(\tau)$ from rising beyond an acceptable level. The discussion above leads to a definition of φ_\pm :

$$\begin{aligned} \varphi_\pm(\psi|\tau_0) &= \int_0^L \psi_{y,\tau_0} K(y, x, \tau_0, \tau_0 + \delta | c_\tau = c_\tau^\pm) dy \\ c_\tau^+ &= 2 \cdot \mathbf{1}_{\{\tau \in [\tau_0, \tau_0 + \frac{\epsilon}{2}]\}} + 3 \cdot \mathbf{1}_{\{\tau \in [\tau_0 + \frac{\epsilon}{2}, \tau_0 + \epsilon]\}} \\ c_\tau^- &= 3 \cdot \mathbf{1}_{\{\tau \in [\tau_0, \tau_0 + \frac{\epsilon}{2}]\}} + 2 \cdot \mathbf{1}_{\{\tau \in [\tau_0 + \frac{\epsilon}{2}, \tau_0 + \epsilon]\}} \end{aligned}$$

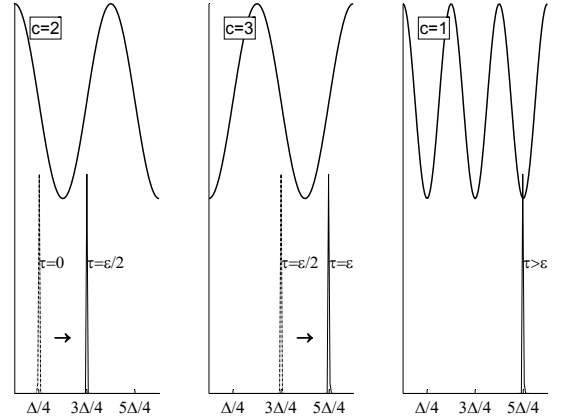


FIG. 2. Graphical explanation of the effects described in sections IIIB 1 and IIIB 2.

Figure 3 (left and central panels) gives a graphical explanation of ϕ_+ .

3. φ_\pm near to the boundaries

The periodic potential in the torus is translation invariant with respect to the transformation $x \mapsto x \pm \Delta$. This implies that the mechanism described in paragraph IIIB 2 holds in any portion of the system in the same way. This leads to eq. (17). Eq. (12) originates from the fact that x_{loc} can be moved only by Δ long steps and so there are only N allowed positions where x_{loc} can be found, as shown in Figure 3 (right panel).

On the other hand, the box boundary condition has no translation invariance and the localized matter wave is reflected by the infinite potential walls. As shown in Figure 6 x_{loc} is shifted by $\Delta/2$ near to the boundaries, because $x_{loc}(\tau/2) = x_{loc}(\tau)$. This fact implies that there are $2N$ allowed positions where x_{loc} can be found. We can enumerate these positions in the order that we obtain by iterative application of $\phi_+(\cdot|\tau)$ operator to ψ_{x,τ_0} . The resulting order is described by eq. (11). An example is shown in Figure 3 (left panel).

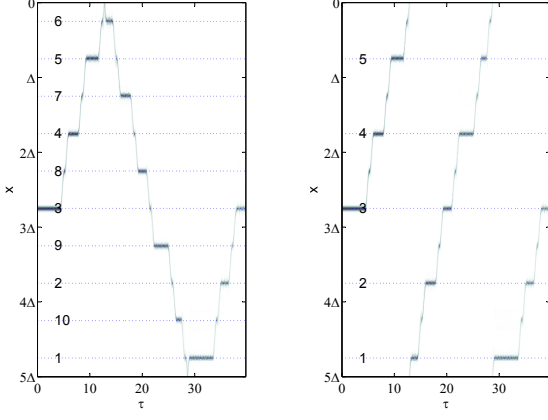


FIG. 3. Density plot of $|\psi(x, \tau)|^2$, using the specifications shown in Figure 1 (periodic case v_1^p). We applied $\varphi_\delta \circ \varphi_+$ ten times, considering a random different δ value per application. The same pattern is simulated considering both box (left panel) and boundary (right panel) conditions.

IV. MULTI-STATE BIT UNDER VARIOUS SETTINGS

In this section we verify the possibility of the system to be used as a multi-state bit under different parameters choices. We compare the results using DPR, defined in section II A. Furthermore, we are interested in finding a stable multi-bit example which can be also feasible in laboratory.

We have compared the two considered potential choices v_1^d and v_1^p , using them to storage information in the system. In case we choose to maintain $c_\tau = 1$ constant, both of them are equally good in keeping the matter wave localized over time. v_1^p results to be better than v_1^d when ϕ_\pm is applied repeatedly to the system. This is the case when we want to write information to be stored in the system. An example can be seen in figure 4 (first row and third row panels). When using v_1^d , $|s_1|$ must be big enough to keep $\psi(x, \tau)$ localized over time, but not big enough to cause fragmentation phenomena. If the matter wave is fragmented, DPR increases when $\psi(x, \tau)$ is forced to oscillate ($c_t > 1$). Figure 5 shows an example of optimal $|s_1|$ level of when using v_1^d . On the other hand, when using v_1^p , s_1 can be chosen arbitrarily high without fragmenting the matter wave. This is the reason why v_1^p leads to a more stable multi-bit behavior than v_1^d .

When considering self-interaction, we observed an increased stability (lower DPR over time) choosing $\beta < 0$, especially if using v_1^p . An example of this fact is shown in figure 4 (second row and fourth row panels). Intuitively, $\beta > 0$ decreases the stability of the system.

Moreover, all the potential amplitudes must be higher at increasing N values, in order to keep $\psi(x, \tau)$ confined in a local fluctuation of the potential when the total number of fluctuations N is bigger. An example of this fact

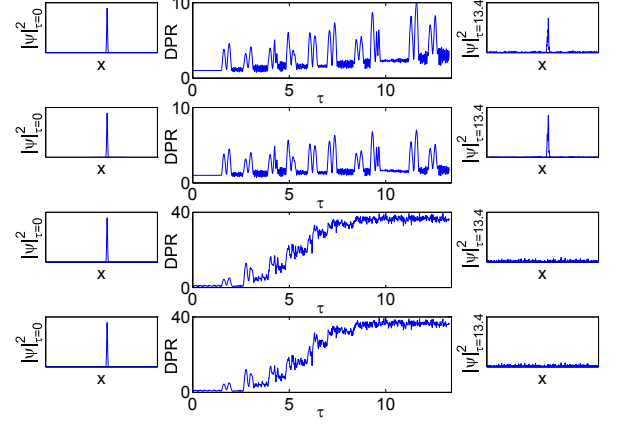


FIG. 4. Time evolution of four different versions of the multi-bit system under the same pattern used in Figure 3. The left panels show the initial density $|\psi(x, \tau = 0)|^2$. The central panels plot $DPR(\tau)$ during the evolution of the system. The right panels show the final state of the system $|\psi(x, \tau = 13.4)|^2$. From the top to the bottom: periodic case (v_1^p), $\alpha = \beta = 0$; periodic case (v_1^p), $\alpha = 0.01\beta = -15$; disordered case (v_1^d), $\alpha = \beta = 0$; disordered case (v_1^d), $\alpha = 0.01$ and $\beta = -15$. All the considered versions have in common the following settings: $N = 5$; $a_i = 120$ ($i = 0, \dots, 3$); box boundary conditions.

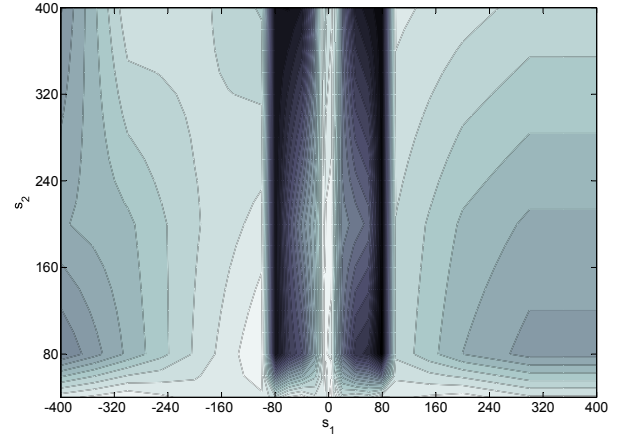


FIG. 5. $DPR[\psi]$ after five applications of $\varphi_\delta \circ \varphi_+$ ($N = 7$; disordered potential v_1^d ; $\alpha = \beta = 0$). Darker areas corresponds to lower DPR values.

is shown in figure 6. Considering these results, a feasible experimental setting that allows to observe a stable ten states multi-bit could be the following: ^{39}K elongated BEC (10^4 atoms) under box boundary conditions; system dimensions $300\mu m \times 30\mu m$; periodic v_1^p potential; $s_1 = s_2 = s_3 \geq 90$; $N = 5$. Please see appendix A (and references therein) for further details.

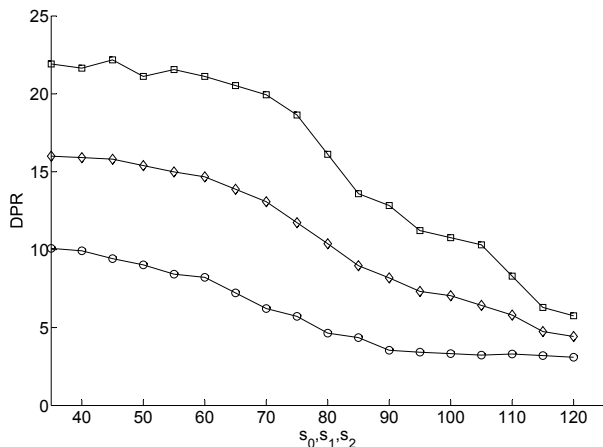


FIG. 6. Final DPR after the evolution of the system ($\alpha = \beta = 0$; box boundary conditions; periodic case v_1^p) using the same pattern described in Figure 3 and applied also in Figure 4. We tested N ranging from 5 (circles) to 7 (squares) and $s_0 = s_1 = s_2$ ranging from 35 to 120.

V. SUMMARY

We have developed a technique to change and preserve the position of a localized matter wave. This behavior is directly applicable to obtain a multi-state memory device. This system can be obtained using optical potentials already available in laboratory. The multi-bit behavior can be observed under multiple parameters choices and we have suggested a fully specified multi-bit which could be realized nowadays. Given that BECs and optical potentials are currently investigated from a quantum information perspective, this work opens the possibility of turning the same BEC from a q-bit into a classical multi-state bit and vice versa in the future.

VI. ACKNOWLEDGEMENTS

Fruitful discussions with M. Modugno are acknowledged.

Appendix A: 1D NPSE in our units

We consider the 1D NPSE equation [35], which describe the dynamics of an elongated BEC:

$$i\hbar \frac{\partial}{\partial t} \psi = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V + \frac{gN_a}{2\pi a_\perp} \frac{|\psi|^2}{\sqrt{1 + 2a_s N_a |\psi|^2}} \right] \psi, \quad (A1)$$

$$+ \frac{\hbar\omega_\perp}{2} \left(\frac{1}{\sqrt{1 + 2a_s N_a |\psi|^2}} + \sqrt{1 + 2a_s N_a |\psi|^2} \right) \psi,$$

with $a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}$ and $g = \frac{4\pi\hbar^2 a_s}{m}$. Let us introduce the following quantities

$$\tau := \frac{E_\xi t}{\hbar} \quad (A2)$$

$$\alpha := \frac{\hbar\omega_\perp}{2} \frac{1}{E_\xi} = \frac{\xi^2}{a_\perp^2} \quad (A3)$$

$$\beta := 2a_s N_a \quad (A4)$$

Moreover, it holds that

$$\frac{gN_a}{2\pi a_\perp} \frac{1}{E_\xi} = \frac{4\pi\hbar^2 a_s}{m} \frac{N_a}{2\pi a_\perp} \frac{2m\xi^2}{\hbar^2} = 2\alpha\beta \quad (A5)$$

multiplying Eq. (A2) by $\frac{1}{E_\xi}$ and replacing eq. (A3, A4, A5) and choosing $\xi = 1$ as the spatial unit, we have

$$i \frac{\partial}{\partial \tau} \psi = \left[-\frac{\partial^2}{\partial x^2} + v + \alpha \left(\frac{2\beta|\psi|^2 + 1}{\sqrt{1 + \beta|\psi|^2}} + \sqrt{1 + \beta|\psi|^2} \right) \right] \psi. \quad (A6)$$

We simulate a ^{39}K condensate with tunable attractive interactions. The following parameters values are accessible to the experiments (see [23] and [32] amongst others): $\xi \simeq 1\mu\text{m}$, $N_a \simeq 10^4$, $L \simeq 300\xi$, $a_\perp \simeq 30\xi$, $0 \geq a_s \gtrsim -7.5 \cdot 10^{-4}\xi$. This leads to $\alpha \simeq 10^{-2}$ and $\beta \in [-15, 0]\xi$.

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- [1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman e E. A. Cornell, *Science* **269**, 198 (1995).
 - [2] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).
 - [3] D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Phys. Rev. Lett.* **77**, 420 (1996).
 - [4] M. Inguscio, *Science* **292**, 452 (2001).
 - [5] C. Fort, F. Minardi, M. Modugno, M. Inguscio, *Recent advances in metrology and fundamental constants* **146**, 765 (2001).
 - [6] F. Ferlaino, P. Maddaloni, S. Burger, F.S. Cataliotti, C. Fort, M. Modugno, M. Inguscio, *Phys. Rev. A* **66**, 011604

- (2002).
- [7] K. Henderson, C. Ryu, C. MacCormick, M.G. Boshier, *New J. Phys* **11** (2009) 043030
- [8] F.K. Abdullaev, R.M. Galimzyanov, K.N. Ismatullaev, *J. Phys. B* **41** 015301 (2008)
- [9] M. D. Girardeau, E. M. Wright, J. M. Triscari *Phys. Rev. A* **63**, 033601 (2001)
- [10] Xiao-Fei Zhang, Qin Yang, Jie-Fang Zhang, X. Z. Chen, W. M. Liu *Phys. Rev. A* **77**, 023613 (2008)
- [11] F.S. Cataliotti, L. Fallani, F. Ferlaino, C. Fort, P. Maddaloni, M. Inguscio, *J. Opt. B* **5**, 571 (2003)
- [12] C. Fort, F.S. Cataliotti, L. Fallani, F. Ferlaino, P. Maddaloni, M. Inguscio, *Phys. Rev. Lett.* **90**, 140405 (2003).

- [13] L. Fallani, L. De Sarlo, J.E. Lye, M. Modugno, R. Saers, C. Fort, M. Inguscio, Phys. Rev. Lett. **93**, 140406 (2004).
- [14] F. Ferlaino, E. De Mirandes, R. Heidemann, G. Roati, G. Modugno, M. Inguscio, *Journal de Physique IV* **116**, 253 (2004).
- [15] T. P. Meyrath, F. Schreck, J. L. Hanssen, C. S. Chu, M. G. Raizen, Phys. Rev. A **71**, 041604(R) (2005).
- [16] L. Fallani, C. Fort, M. Inguscio, Adv. At. Mol. Opt. Phys. **56**, 119 (2008).
- [17] B. Shapiro, J. Phys. A **45** 143001 (2012)
- [18] L. Sanchez-Palencia, M. Lewenstein, Nat. Phys. **6**, 87–95 (2010)
- [19] G. Modugno Rep. Prog. Phys. **73** 102401 (2010)
- [20] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, M. Lewenstein, Phys. Rev. Lett. **91**, 080403 (2003).
- [21] J. E. Lye, L. Fallani, M. Modugno, D. S. Wiersma, C. Fort, M. Inguscio, Phys. Rev. Lett. **95**, 070401 (2005).
- [22] C. Fort, L. Fallani, V. Guarrera, J. E. Lye, M. Modugno, D. S. Wiersma, M. Inguscio, Phys. Rev. Lett. **95**, 170410 (2005).
- [23] D. Clement, A.F. Varon, J.A. Retter, L. Sanchez-Palencia, A. Aspect, P. Bouyer, New J. Phys. **8** 165 (2006).
- [24] M. Modugno, Phys. Rev. A **73**, 013606 (2006).
- [25] G. M. Falco, A. A. Fedorenko, J. Giacomelli, M. Modugno, Phys. Rev. A **82**, 053405 (2010).
- [26] J. Giacomelli, Physica A **404** 158 (2014)
- [27] J. W. Goodman, *Speckle Phenomena in Optics: Theory and Applications*, (Roberts, 2005).
- [28] B. Wu, Q. Niu, Phys. Rev. A **64**, 061603(R) (2001).
- [29] A. Smerzi, A. Trombettoni, P. G. Kevrekidis, and A. R. Bishop, Phys. Rev. Lett. **89**, 170402 (2002).
- [30] L. Khaykovich *et al.*, Science **296**, 1290 (2002);
- [31] K. E. Strecker *et al.*, Nature London **417**, 150 (2002).
- [32] G. Roati, M. Zaccanti, C. D'Errico, J. Catani, M. Modugno, A. Simoni, M. Inguscio, G. Modugno, Phys. Rev. Lett. **99**, 010403 (2007).
- [33] A. Ramanathan, K.C. Wright, S.R. Muniz, M. Zelan, W.T. Hill, C.J. Lobb, K. Helmerson, W.D. Phillips, G.K. Campbell, Phys. Rev. Lett. **106**, 130401 (2011)
- [34] S. Stringari and L. Pitaevskii, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).
- [35] L. Salasnich, A. Parola, L. Reatto, Phys. Rev. A **65**, 043614 (2002).
- [36] V. Ahufinger, A. Mebrahtu, R. Corbalan, A. Sanpera, New J. Phys. **9** 4 (2007)
- [37] Z.M. Wang, L.A. Wu, M. Modugno, M. S. Byrd, T. Yu, J. Q. You, Phys. Rev. A **89** 042326 (2014)
- [38] F. Evers, A.D. Mirlin, Phys. Rev. Lett. **84**, 3690 (2000)